

On a conjecture of Anosov

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Abstract

In this paper, we prove that for every bumpy Finsler n -sphere (S^n, F) with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, there exist $2[\frac{n+1}{2}]$ prime closed geodesics. This gives a confirmed answer to a conjecture of D. V. Anosov [Ano] in 1974 for a generic case.

Key words: Finsler spheres, closed geodesics, index iteration, mean index identity, stability.

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1 Introduction and main results

This paper is devoted to a study on closed geodesics on Finsler spheres. Let us recall firstly the definition of the Finsler metrics.

Definition 1.1. (cf. [She]) *Let M be a finite dimensional manifold. A function $F : TM \rightarrow [0, +\infty)$ is a Finsler metric if it satisfies*

(F1) F is C^∞ on $TM \setminus \{0\}$,

(F2) $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in T_x M$, $x \in M$, and $\lambda > 0$,

(F3) For every $y \in T_x M \setminus \{0\}$, the quadratic form

$$g_{x,y}(u, v) \equiv \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t=s=0}, \quad \forall u, v \in T_x M,$$

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is positive definite.

In this case, (M, F) is called a Finsler manifold. F is reversible if $F(x, -y) = F(x, y)$ holds for all $y \in T_x M$ and $x \in M$. F is Riemannian if $F(x, y)^2 = \frac{1}{2}G(x)y \cdot y$ for some symmetric positive definite matrix function $G(x) \in GL(T_x M)$ depending on $x \in M$ smoothly.

A closed curve in a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [She]). As usual, on any Finsler n -sphere $S^n = (S^n, F)$, a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \rightarrow S^n$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m -th iteration c^m of c is defined by $c^m(t) = c(mt)$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1 - t)$ for $t \in \mathbf{R}$. Note that on a non-symmetric Finsler manifold, the inverse curve of a closed geodesic is not a closed geodesic in general. We call two prime closed geodesics c and d *distinct* if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbf{R}$. We shall omit the word *distinct* when we talk about more than one prime closed geodesic. On a symmetric Finsler (or Riemannian) n -sphere, two closed geodesics c and d are called *geometrically distinct* if $c(S^1) \neq d(S^1)$, i.e., their image sets in S^n are distinct.

For a closed geodesic c on (S^n, F) , denote by P_c the linearized Poincaré map of c . Then $P_c \in \text{Sp}(2n - 2)$ is symplectic. For any $M \in \text{Sp}(2k)$, we define the *elliptic height* $e(M)$ of M to be the total algebraic multiplicity of all eigenvalues of M on the unit circle $\mathbf{U} = \{z \in \mathbf{C} \mid |z| = 1\}$ in the complex plane \mathbf{C} . Since M is symplectic, $e(M)$ is even and $0 \leq e(M) \leq 2k$. A closed geodesic c is called *elliptic* if $e(P_c) = 2(n - 1)$, i.e., all the eigenvalues of P_c locate on \mathbf{U} ; *hyperbolic* if $e(P_c) = 0$, i.e., all the eigenvalues of P_c locate away from \mathbf{U} ; *non-degenerate* if 1 is not an eigenvalue of P_c . A Finsler sphere (S^n, F) is called *bumpy* if all the closed geodesics on it are non-degenerate.

Following H-B. Rademacher in [Rad3], the reversibility $\lambda = \lambda(M, F)$ of a compact Finsler manifold (M, F) is defined to be

$$\lambda := \max\{F(-X) \mid X \in TM, F(X) = 1\} \geq 1.$$

It was quite surprising when Katok [Kat] in 1973 found some non-reversible Finsler metrics on CROSS with only finitely many prime closed geodesics and all closed geodesics are non-degenerate and elliptic. The smallest number of closed geodesics on S^n that one obtains in these examples is $2[\frac{n+1}{2}]$ (cf. [Zil]). Then D. V. Anosov in I.C.M. of 1974 conjectured that the lower bound of the number of closed geodesics on any Finsler sphere (S^n, F) should be $2[\frac{n+1}{2}]$, i.e., the number of closed geodesics in Katok's example.

We can show that under some conditions this conjecture is true, i.e., the following main result of the paper.

Theorem 1.2. *For every bumpy Finsler n -sphere (S^n, F) with reversibility λ and flag curvature K satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, there exist $2\lfloor \frac{n+1}{2} \rfloor$ prime closed geodesics.*

We can obtain a stability result.

Theorem 1.3. *For every bumpy Finsler n -sphere (S^n, F) with reversibility λ and flag curvature K satisfying $\left(\frac{3\lambda}{2(\lambda+1)}\right)^2 < K \leq 1$, there exist two elliptic prime closed geodesics provided the number of prime closed geodesics on (S^n, F) is finite.*

Remark 1.4. In [BTZ2], W. Ballmann, G. Thorbergsson and W. Ziller proved that for a Riemannian metric on S^n with sectional curvature $1/4 \leq K \leq 1$ there exist $g(n)$ geometrically distinct closed geodesics, and $\frac{n(n+1)}{2}$ geometrically distinct closed geodesics if the metric is bumpy. In [BaL], V. Bangert and Y. Long proved that on any Finsler 2-sphere (S^2, F) , there exist at least two prime closed geodesics, which solves Anosov's conjecture for the S^2 case.. In [LoW2] of Y. Long and the author, they further proved the existence of at least two irrationally elliptic prime closed geodesics on every Finsler 2-sphere (S^2, F) provided the number of prime closed geodesics is finite. In [Rad4], H.-B. Rademacher studied the existence and stability of closed geodesics on positively curved Finsler manifolds. In a series papers [LoD], [DuL1]-[DuL3] of Y. Long and H. Duan, they proved there exist two prime closed geodesics on any compact simply connected Finsler or Riemannian manifold. In [Rad5], H.-B. Rademacher proved there exist two prime closed geodesics on any bumpy n -sphere. In [Wang], the author proved there exist three prime closed geodesics on any (S^3, F) satisfying $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$.

Our proof of these theorems contains mainly three ingredients: the common index jump theorem of Y. Long, Morse theory and the mean index identity of H.-B. Rademacher. Fix a Finsler metric F on S^n . Let $\Lambda = \Lambda S^n$ be the free loop space of S^n , which is a Hilbert manifold. For definition and basic properties of Λ , we refer readers to [Kli2] and [Kli3]. Let $E(c) = \frac{1}{2} \int_0^1 F(\dot{c}(t))^2 dt$ be the energy functional on Λ . In this paper for $\kappa \in \mathbf{R}$ we denote by

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \leq \kappa\}, \quad (1.1)$$

and consider the quotient space Λ/S^1 . Since the energy functional E is S^1 -invariant, the negative gradient flow of E induce a flow on Λ/S^1 , so we can apply Morse theory on Λ/S^1 . By a result of H.-B. Rademacher in [Rad1] of 1989, we get the Morse series of the space pair $(\Lambda/S^1, \Lambda^0/S^1)$ with rational coefficients. The reason we use $(\Lambda/S^1, \Lambda^0/S^1)$ instead of (Λ, Λ^0) is that the Morse series of the first is lacunary.

In this paper, let \mathbf{N} , \mathbf{N}_0 , \mathbf{Z} , \mathbf{Q} , \mathbf{R} , and \mathbf{C} denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only

singular homology modules with \mathbf{Q} -coefficients. For terminologies in algebraic topology we refer to [GrH]. For $k \in \mathbf{N}$, we denote by \mathbf{Q}^k the direct sum $\mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$ of k copies of \mathbf{Q} and $\mathbf{Q}^0 = 0$. For an S^1 -space X , we denote by \overline{X} the quotient space X/S^1 . We define the functions

$$\begin{cases} [a] = \max\{k \in \mathbf{Z} \mid k \leq a\}, & \mathcal{E}(a) = \min\{k \in \mathbf{Z} \mid k \geq a\}, \\ \varphi(a) = \mathcal{E}(a) - [a], & \{a\} = a - [a]. \end{cases} \quad (1.2)$$

Especially, $\varphi(a) = 0$ if $a \in \mathbf{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbf{Z}$.

2 Variational structures of closed geodesics

In this section, we review the variational structures of closed geodesic, all the details can be found in [Rad2] or [BaL].

On a compact Finsler manifold (M, F) , we choose an auxiliary Riemannian metric. This endows the space $\Lambda = \Lambda M$ of H^1 -maps $\gamma : S^1 \rightarrow M$ with a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbf{R}/\mathbf{Z}$ acts continuously by isometries, cf. [Kli2], Chapters 1 and 2. This action is defined by translating the parameter, i.e.

$$(s \cdot \gamma)(t) = \gamma(t + s)$$

for all $\gamma \in \Lambda$ and $s, t \in S^1$. The Finsler metric F defines an energy functional E and a length functional L on Λ by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\dot{\gamma}(t))^2 dt, \quad L(\gamma) = \int_{S^1} F(\dot{\gamma}(t)) dt. \quad (2.1)$$

Both functionals are invariant under the S^1 -action. The critical points of E of positive energies are precisely the closed geodesics $c : S^1 \rightarrow M$ of the Finsler structure. If $c \in \Lambda$ is a closed geodesic then c is a regular curve, i.e. $\dot{c}(t) \neq 0$ for all $t \in S^1$, and this implies that the second differential $E''(c)$ of E at c exists. As usual we define the index $i(c)$ of c as the maximal dimension of subspaces of $T_c \Lambda$ on which $E''(c)$ is negative definite, and the nullity $\nu(c)$ of c so that $\nu(c) + 1$ is the dimension of the null space of $E''(c)$.

For $m \in \mathbf{N}$ we denote the m -fold iteration map $\phi^m : \Lambda \rightarrow \Lambda$ by

$$\phi^m(\gamma)(t) = \gamma(mt) \quad \forall \gamma \in \Lambda, t \in S^1. \quad (2.2)$$

We also use the notation $\phi^m(\gamma) = \gamma^m$. For a closed geodesic c , the mean index is defined to be:

$$\hat{i}(c) = \lim_{m \rightarrow \infty} \frac{i(c^m)}{m}. \quad (2.3)$$

If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of γ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. If $m(\gamma) = 1$ then γ is called *prime*. Hence $m(\gamma) = m$ if and only if there exists a prime curve $\tilde{\gamma} \in \Lambda$ such that $\gamma = \tilde{\gamma}^m$.

For a closed geodesic c we set

$$\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}.$$

If $A \subseteq \Lambda$ is invariant under some subgroup Γ of S^1 , we denote by A/Γ the quotient space of A with respect to the action of Γ .

Using singular homology with rational coefficients we will consider the following critical \mathbf{Q} -module of a closed geodesic $c \in \Lambda$:

$$\overline{\mathcal{C}}_*(E, c) = H_* \left((\Lambda(c) \cup S^1 \cdot c) / S^1, \Lambda(c) / S^1 \right). \quad (2.4)$$

We call a closed geodesic satisfying the isolation condition, if the following holds:

(Iso) For all $m \in \mathbf{N}$ the orbit $S^1 \cdot c^m$ is an isolated critical orbit of E .

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all the closed geodesics satisfy (Iso).

The following Propositions were proved in [Rad2] and [BaL].

Proposition 2.1. (cf. Satz 6.11 of [Rad2] or Proposition 3.12 of [BaL]) *Let c be a prime closed geodesic on a bumpy Finsler manifold (M, F) satisfying (Iso). Then we have*

$$\overline{\mathcal{C}}_q(E, c^m) = \begin{cases} \mathbf{Q}, & \text{if } i(c^m) - i(c) \in 2\mathbf{Z}, \text{ and } q = i(c^m) \\ 0, & \text{otherwise.} \end{cases} \quad (2.5)$$

Now we briefly describe the relative homological structure of the quotient space $\overline{\Lambda} \equiv \overline{\Lambda} S^n$. Here we have $\overline{\Lambda}^0 = \overline{\Lambda}^0 S^n = \{\text{constant point curves in } S^n\} \cong S^n$.

Theorem 2.2. (H.-B. Rademacher, Theorem 2.4 and Remark 2.5 of [Rad1]) *We have the Poincaré series*

(i) *When $n = 2k + 1$ is odd*

$$\begin{aligned} P(\overline{\Lambda} S^n, \overline{\Lambda}^0 S^n)(t) &= t^{n-1} \left(\frac{1}{1-t^2} + \frac{t^{n-1}}{1-t^{n-1}} \right) \\ &= t^{2k} \left(\frac{1}{1-t^2} + \frac{t^{2k}}{1-t^{2k}} \right). \end{aligned} \quad (2.6)$$

Thus for $q \in \mathbf{Z}$ and $l \in \mathbf{N}_0$, we have

$$\begin{aligned}
b_q &= b_q(\overline{\Lambda}S^n, \overline{\Lambda}^0S^n) \\
&= \text{rank}H_q(\overline{\Lambda}S^n, \overline{\Lambda}^0S^n) \\
&= \begin{cases} 2, & \text{if } q \in \{4k + 2l, \quad l = 0 \bmod k\}, \\ 1, & \text{if } q \in \{2k\} \cup \{2k + 2l, \quad l \neq 0 \bmod k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}
\end{aligned}$$

(ii) When $n = 2k$ is even

$$\begin{aligned}
P(\overline{\Lambda}S^n, \overline{\Lambda}^0S^n)(t) &= t^{n-1} \left(\frac{1}{1-t^2} + \frac{t^{n(m+1)-2}}{1-t^{n(m+1)-2}} \right) \frac{1-t^{nm}}{1-t^n} \\
&= t^{2k-1} \left(\frac{1}{1-t^2} + \frac{t^{4k-2}}{1-t^{4k-2}} \right), \tag{2.8}
\end{aligned}$$

where $m = 1$ by Theorem 2.4 of [Rad1]. Thus for $q \in \mathbf{Z}$ and $l \in \mathbf{N}_0$, we have

$$\begin{aligned}
b_q &= b_q(\overline{\Lambda}S^n, \overline{\Lambda}^0S^n) \\
&= \text{rank}H_q(\overline{\Lambda}S^n, \overline{\Lambda}^0S^n) \\
&= \begin{cases} 2, & \text{if } q \in \{6k - 3 + 2l, \quad l = 0 \bmod 2k - 1\}, \\ 1, & \text{if } q \in \{2k - 1\} \cup \{2k - 1 + 2l, \quad l \neq 0 \bmod 2k - 1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}
\end{aligned}$$

We have the following version of the Morse inequality.

Theorem 2.3. (Theorem 6.1 of [Rad2]) *Suppose that there exist only finitely many prime closed geodesics $\{c_j\}_{1 \leq j \leq p}$ on (M, F) , and $0 \leq a < b \leq \infty$ are regular values of the energy functional E . Define for each $q \in \mathbf{Z}$,*

$$\begin{aligned}
M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) &= \sum_{1 \leq j \leq p, a < E(c_j^m) < b} \text{rank} \overline{C}_q(E, c_j^m) \\
b_q(\overline{\Lambda}^b, \overline{\Lambda}^a) &= \text{rank}H_q(\overline{\Lambda}^b, \overline{\Lambda}^a).
\end{aligned}$$

Then there holds

$$\begin{aligned}
M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) &= M_{q-1}(\overline{\Lambda}^b, \overline{\Lambda}^a) + \cdots + (-1)^q M_0(\overline{\Lambda}^b, \overline{\Lambda}^a) \\
&\geq b_q(\overline{\Lambda}^b, \overline{\Lambda}^a) - b_{q-1}(\overline{\Lambda}^b, \overline{\Lambda}^a) + \cdots + (-1)^q b_0(\overline{\Lambda}^b, \overline{\Lambda}^a), \tag{2.10}
\end{aligned}$$

$$M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) \geq b_q(\overline{\Lambda}^b, \overline{\Lambda}^a). \tag{2.11}$$

3 Classification of closed geodesics on S^n

Let c be a closed geodesic on a Finsler n -sphere $S^n = (S^n, F)$. Denote the linearized Poincaré map of c by $P_c \in \text{Sp}(2n - 2)$. Then P_c is a symplectic matrix. Note that the index iteration formulae in [Lon3] of 2000 (cf. Chap. 8 of [Lon4]) work for Morse indices of iterated closed geodesics (cf. [LiL], Chap. 12 of [Lon4]). Since every closed geodesic on a sphere must be orientable. Then by Theorem 1.1 of [Liu] of C. Liu (cf. also [Wil]), the initial Morse index of a closed geodesic c on a n -dimensional Finsler sphere coincides with the index of a corresponding symplectic path introduced by C. Conley, E. Zehnder, and Y. Long in 1984-1990 (cf. [Lon4]).

As in §1.8 of [Lon4], define the homotopy component $\Omega^0(P_c)$ of P_c to be the path component of $\Omega(P_c)$, where

$$\begin{aligned} \Omega(P_c) = \{N \in \text{Sp}(2n - 2) \mid & \sigma(N) \cap U = \sigma(P_c) \cap U, \text{ and} \\ & \nu_\lambda(N) = \nu_\lambda(P_c) \ \forall \lambda \in \sigma(P_c) \cap U\}. \end{aligned} \quad (3.1)$$

The next theorem is due to Y. Long (cf. Theorem 8.3.1 and Corollary 8.3.2 of [Lon4]).

Theorem 3.1. *Let $\gamma \in \{\xi \in C([0, \tau], \text{Sp}(2n)) \mid \xi(0) = I\}$, Then there exists a path $f \in C([0, 1], \Omega^0(\gamma(\tau)))$ such that $f(0) = \gamma(\tau)$ and*

$$\begin{aligned} f(1) = & N_1(1, 1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1, -1)^{\diamond p_+} \diamond N_1(-1, 1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1, -1)^{\diamond q_+} \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \diamond N_2(\omega_1, u_1) \diamond \cdots \diamond N_2(\omega_{r_*}, u_{r_*}) \\ & \diamond N_2(\lambda_1, v_1) \diamond \cdots \diamond N_2(\lambda_{r_0}, v_{r_0}) \diamond M_0 \end{aligned} \quad (3.2)$$

where $N_2(\omega_j, u_j)$ s are non-trivial and $N_2(\lambda_j, v_j)$ s are trivial basic normal forms; $\sigma(M_0) \cap U = \emptyset$; $p_-, p_0, p_+, q_-, q_0, q_+, r, r_*$ and r_0 are non-negative integers; $\omega_j = e^{\sqrt{-1}\alpha_j}$, $\lambda_j = e^{\sqrt{-1}\beta_j}$; $\theta_j, \alpha_j, \beta_j \in (0, \pi) \cup (\pi, 2\pi)$; these integers and real numbers are uniquely determined by $\gamma(\tau)$. Then using the functions defined in (1.2).

$$\begin{aligned} i(\gamma, m) = & m(i(\gamma, 1) + p_- + p_0 - r) + 2 \sum_{j=1}^r \mathcal{E}\left(\frac{m\theta_j}{2\pi}\right) - r - p_- - p_0 \\ & - \frac{1 + (-1)^m}{2}(q_0 + q_+) + 2 \left(\sum_{j=1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) - r_* \right), \end{aligned} \quad (3.3)$$

$$\begin{aligned} \nu(\gamma, m) = & \nu(\gamma, 1) + \frac{1 + (-1)^m}{2}(q_- + 2q_0 + q_+) + 2(r + r_* + r_0) \\ & - 2 \left(\sum_{j=1}^r \varphi\left(\frac{m\theta_j}{2\pi}\right) + \sum_{j=1}^{r_*} \varphi\left(\frac{m\alpha_j}{2\pi}\right) + \sum_{j=1}^{r_0} \varphi\left(\frac{m\beta_j}{2\pi}\right) \right), \end{aligned} \quad (3.4)$$

$$\hat{i}(\gamma, 1) = i(\gamma, 1) + p_- + p_0 - r + \sum_{j=1}^r \frac{\theta_j}{\pi}. \quad (3.5)$$

Where $N_1(1, \pm 1) = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, $N_1(-1, \pm 1) = \begin{pmatrix} -1 & \pm 1 \\ 0 & -1 \end{pmatrix}$, $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, $N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbf{R}^{2 \times 2}$, such that $(b_2 - b_3) \sin \theta > 0$, if $N_2(\omega, b)$ is trivial; $(b_2 - b_3) \sin \theta < 0$, if $N_2(\omega, b)$ is non-trivial. We have $i(\gamma, 1)$ is odd if $f(1) = N_1(1, 1)$, I_2 , $N_1(-1, 1)$, $-I_2$, $N_1(-1, -1)$ and $R(\theta)$; $i(\gamma, 1)$ is even if $f(1) = N_1(1, -1)$ and $N_2(\omega, b)$; $i(\gamma, 1)$ can be any integer if $\sigma(f(1)) \cap \mathbf{U} = \emptyset$.

4 A mean index equality on (S^n, F)

In this section, we recall the mean index equality obtained in [Rad1]. Suppose that there are only finitely many prime closed geodesics $\{c_j\}_{1 \leq j \leq p}$ on a bumpy (S^n, F) with $\hat{i}(c_j) > 0$ for $1 \leq j \leq p$.

Lemma 4.1. *Let c be a prime closed geodesic on a bumpy (S^n, F) . Then we have*

$$i(c^{p+2}) - i(c^p) \in 2\mathbf{Z}, \quad \forall p \in \mathbf{N}. \quad (4.1)$$

Proof. This follows directly from Theorem 3.1. ■

Definition 4.2. *Suppose c is a closed geodesic on (S^n, F) . The Euler characteristic $\chi(c^m)$ of c^m is defined by*

$$\begin{aligned} \chi(c^m) &\equiv \chi\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right), \\ &\equiv \sum_{q=0}^{\infty} (-1)^q \dim \overline{C}_q(E, c^m). \end{aligned} \quad (4.2)$$

Here $\chi(A, B)$ denotes the usual Euler characteristic of the space pair (A, B) .

The average Euler characteristic $\hat{\chi}(c)$ of c is defined by

$$\hat{\chi}(c) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq m \leq N} \chi(c^m). \quad (4.3)$$

The following remark shows that $\hat{\chi}(c)$ is well-defined and is a rational number.

Remark 4.3. By Proposition 2.1 and Lemma 4.1 we have

$$\begin{aligned} \hat{\chi}(c) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq m \leq N} (-1)^{i(c^m)} \dim \overline{C}_{i(c^m)}(E, c^m) \\ &= \lim_{s \rightarrow \infty} \frac{1}{2s} \sum_{\substack{1 \leq m \leq 2 \\ 0 \leq p < s}} (-1)^{i(c^{2p+m})} \dim \overline{C}_{i(c^{2p+m})}(E, c^m) \\ &= \frac{1}{2} \sum_{1 \leq m \leq 2} (-1)^{i(c^m)} \dim \overline{C}_{i(c^m)}(E, c^m) = \frac{1}{2} \sum_{1 \leq m \leq 2} \chi(c^m). \end{aligned} \quad (4.4)$$

Therefore $\hat{\chi}(c)$ is well defined and is a rational number.

The following is the mean index equality of H.-B. Rademacher (Theorem 7.9 in [Rad2]).

Theorem 4.4. *Suppose that there exist only finitely many prime closed geodesics $\{c_j\}_{1 \leq j \leq p}$ with $\hat{i}(c_j) > 0$ for $1 \leq j \leq p$ on (S^n, F) . Then the following identity holds*

$$\sum_{1 \leq j \leq p} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(n, 1) = \begin{cases} \frac{n+1}{2(n-1)}, & n \text{ odd}, \\ \frac{-n}{2(n-1)}, & n \text{ even}. \end{cases} \quad (4.5)$$

5 Proof of the main theorems

In this section, we give the proofs of Theorems 1.2 and 1.3 by using the mean index identity in Theorem 4.4, Morse inequality and the index iteration theory developed by Y. Long and his coworkers.

In the following for the notation introduced in Section 3 we use specially $M_j = M_j(\overline{\Lambda}S^n, \overline{\Lambda}^0S^n)$ and $b_j = b_j(\overline{\Lambda}S^n, \overline{\Lambda}^0S^n)$ for $j = 0, 1, 2, \dots$

First note that if the flag curvature K of (S^n, F) satisfies $\left(\frac{\lambda}{\lambda+1}\right)^2 < K \leq 1$, then every nonconstant closed geodesic must satisfy

$$i(c) \geq n - 1, \quad (5.1)$$

$$\hat{i}(c) > n - 1, \quad (5.2)$$

where (5.1) follows from Theorem 3 and Lemma 3 of [Rad3], (5.2) follows from Lemma 2 of [Rad4]. Now it follows from Theorem 2.2 of [LoZ] (Theorem 10.2.3 of [Lon4]) that

$$i(c^{m+1}) - i(c^m) \geq i(c) - \frac{e(P_c)}{2} \geq 0, \quad \forall m \in \mathbf{N}. \quad (5.3)$$

Here the last inequality holds by (5.1) and the fact that $e(P_c) \leq 2(n-1)$.

In the rest of this paper, we will assume the following

(F) There are only finitely many prime closed geodesics $\{c_j\}_{1 \leq j \leq p}$ on (S^n, F) .

By (5.2), we can use the common index jump theorem (Theorem 4.3 of [LoZ], Theorem 11.2.1 of [Lon4]) to obtain some $(N, m_1, \dots, m_p) \in \mathbf{N}^{p+1}$ such that

$$i(c_j^{2m_j}) \geq 2N - \frac{e(P_{c_j})}{2} \geq 2N - (n-1), \quad (5.4)$$

$$i(c_j^{2m_j}) \leq 2N + \frac{e(P_{c_j})}{2} \leq 2N + (n-1), \quad (5.5)$$

$$i(c_j^{2m_j-m}) \leq 2N - (i(c_j) + 2S_{P_{c_j}}^+(1) - \nu(c_j)), \quad \forall m \in \mathbf{N}. \quad (5.6)$$

$$i(c_j^{2m_j+m}) \geq 2N + i(c_j), \quad \forall m \in \mathbf{N}, \quad (5.7)$$

where $S_{P_{c_j}}^+(1)$ denote the splitting number of c_j at 1. Since the metric is bumpy, 1 is not an eigenvalue of P_{c_j} for $1 \leq j \leq p$. Thus we have $S_{P_{c_j}}^+(1) = 0$ and $\nu(c_j) = 0$. Hence (5.6) becomes

$$i(c_j^{2m_j-m}) \leq 2N - i(c_j), \quad \forall m \in \mathbf{N}. \quad (5.8)$$

Moreover we have

$$\min \left\{ \left\{ \frac{m_j \theta}{\pi} \right\}, 1 - \left\{ \frac{m_j \theta}{\pi} \right\} \right\} < \delta, \quad (5.9)$$

whenever $e^{\sqrt{-1}\theta} \in \sigma(P_{c_j})$ and δ can be chosen as small as we want. More precisely, by Theorem 4.1 of [LoZ] (in (11.1.10) in Theorem 11.1.1 of [Lon4], with $D_j = \hat{i}(c_j)$, we have

$$m_j = \left(\left\lfloor \frac{N}{M\hat{i}(c_j)} \right\rfloor + \xi_j \right) M, \quad 1 \leq j \leq p, \quad (5.10)$$

where $\xi_j = 0$ or 1 for $1 \leq j \leq p$ and $M \in \mathbf{N}$. By (11.1.20) in Theorem 11.1.1 of [Lon4], for any $\epsilon > 0$, we can choose N and $\{\xi_j\}_{1 \leq j \leq p}$ such that

$$\left| \frac{N}{M\hat{i}(c_j)} - \left\lfloor \frac{N}{M\hat{i}(c_j)} \right\rfloor - \xi_j \right| < \epsilon < \frac{1}{1 + \sum_{1 \leq j \leq p} 4M|\hat{\chi}(c_j)|}, \quad 1 \leq j \leq p. \quad (5.11)$$

Lemma 5.1. *There exists a prime closed geodesic c_{j_0} such that $i(c_{j_0}^{2m_{j_0}}) = 2N + (n-1)$. In particular, we have $\overline{C}_{2N+n-1}(E, c_{j_0}^{2m_{j_0}}) \neq 0$.*

Proof. Suppose the contrary. Then by (5.5), we have

$$i(c_j^{2m_j}) < 2N + (n-1), \quad 1 \leq j \leq p. \quad (5.12)$$

Now by (5.1), (5.3), (5.7) and (5.12), we have

$$i(c_j^m) \leq i(c_j^{2m_j}), \quad \forall m < 2m_j, \quad (5.13)$$

$$i(c_j^{2m_j}) \leq 2N + n - 2, \quad (5.14)$$

$$i(c_j^m) \geq 2N + n - 1, \quad \forall m > 2m_j. \quad (5.15)$$

By Theorem 4.4, we have

$$\sum_{1 \leq j \leq p} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(n, 1) \in \mathbf{Q}. \quad (5.16)$$

Note by the proof of Theorem 4.1 of [LoZ] (Theorem 11.1.1 of [Lon4]), we can require that $N \in \mathbf{N}$ further satisfies (cf. (11.1.22) in [Lon4])

$$2NB(n, 1) \in \mathbf{Z}. \quad (5.17)$$

Multiplying both sides of (5.16) by $2N$ yields

$$\sum_{1 \leq j \leq p} \frac{2N\hat{\chi}(c_j)}{\hat{i}(c_j)} = 2NB(n, 1). \quad (5.18)$$

Claim 1. *We have*

$$\sum_{1 \leq j \leq p} 2m_j \hat{\chi}(c_j) = 2NB(n, 1). \quad (5.19)$$

In fact, by (5.10) and (5.18), we have

$$\begin{aligned} & 2NB(n, 1) \\ = & \sum_{1 \leq j \leq p} \frac{2N\hat{\chi}(c_j)}{\hat{i}(c_j)} \\ = & \sum_{1 \leq j \leq p} 2\hat{\chi}(c_j) \left(\left\lfloor \frac{N}{M\hat{i}(c_j)} \right\rfloor + \xi_j \right) M + \sum_{1 \leq j \leq p} 2\hat{\chi}(c_j) \left(\frac{N}{M\hat{i}(c_j)} - \left\lfloor \frac{N}{M\hat{i}(c_j)} \right\rfloor - \xi_j \right) M \\ = & \sum_{1 \leq j \leq p} 2m_j \hat{\chi}(c_j) + \sum_{1 \leq j \leq p} 2M\hat{\chi}(c_j)\epsilon_j. \end{aligned} \quad (5.20)$$

By (4.4) we have

$$2m_j \hat{\chi}(c_j) \in \mathbf{Z}, \quad 1 \leq j \leq p. \quad (5.21)$$

Now Claim 1 follows by (5.11), (5.17), (5.20) and (5.21).

Claim 2. *We have*

$$\sum_{1 \leq j \leq p} 2m_j \hat{\chi}(c_j) = M_0 - M_1 + M_2 - \cdots + (-1)^{2N+n-2} M_{2N+n-2}, \quad (5.22)$$

where $M_q \equiv M_q(\overline{\Lambda}, \overline{\Lambda}^0)$ for $q \in \mathbf{Z}$.

In fact, by definition, the right hand side of (5.22) is

$$RHS = \sum_{\substack{q \leq 2N+n-2 \\ m \geq 1, 1 \leq j \leq p}} (-1)^q \dim \overline{C}_q(E, c_j^m). \quad (5.23)$$

By (5.13)-(5.15) and Proposition 2.1, we have

$$RHS = \sum_{\substack{1 \leq j \leq p, 1 \leq m \leq 2m_j \\ q \leq 2N+n-2}} (-1)^q \dim \overline{C}_q(E, c_j^m), \quad (5.24)$$

$$= \sum_{1 \leq j \leq p, 1 \leq m \leq 2m_j} \chi(c_j^m), \quad (5.25)$$

where the second equality follows from (4.2).

By (4.4), we have

$$\begin{aligned}
\sum_{1 \leq m \leq 2m_j} \chi(c_j^m) &= \sum_{0 \leq s < m_j, 1 \leq m \leq 2} \chi(c_j^{2s+m}) \\
&= m_j \sum_{1 \leq m \leq 2} \chi(c_j^m) \\
&= 2m_j \hat{\chi}(c_j),
\end{aligned} \tag{5.26}$$

This proves Claim 2.

Now we consider the following two cases according to the parity of n .

Case 1. $n = 2k + 1$ is odd.

In this case, we have by (4.5)

$$B(n, 1) = \frac{n+1}{2(n-1)} = \frac{k+1}{2k}. \tag{5.27}$$

By the proof of Theorem 4.1 of [LoZ] (Theorem 11.1.1 of [Lon4]), we may further assume $N = 2ks$ for some $s \in \mathbf{N}$.

Thus by (5.19), (5.22) and (5.27), we have

$$M_0 - M_1 + M_2 - \cdots + (-1)^{2N+n-2} M_{2N+n-2} = 2s(k+1). \tag{5.28}$$

On the other hand, we have by (2.7)

$$\begin{aligned}
&b_0 - b_1 + b_2 - \cdots + (-1)^{2N+n-2} b_{2N+n-2} \\
&= b_{2k} + (b_{2k+2} + \cdots + b_{4k} + \cdots + b_{4sk+2} + \cdots + b_{4sk+2k}) - b_{4sk+2k} \\
&= 1 + 2s(k-1+2) - 2 \\
&= 2s(k+1) - 1.
\end{aligned} \tag{5.29}$$

In fact, we cut off the sequence $\{b_{2k+2}, \dots, b_{4sk+2k}\}$ into $2s$ pieces, each of them contains k terms. Moreover, each piece contain 1 for $k-1$ times and 2 for one time. Thus (5.29) holds.

Now by (5.28), (5.29) and Theorem 2.3, we have

$$\begin{aligned}
-2s(k+1) &= M_{2N+n-2} - M_{2N+n-3} + \cdots + M_1 - M_0 \\
&\geq b_{2N+n-2} - b_{2N+n-3} + \cdots + b_1 - b_0 \\
&= -(2s(k+1) - 1).
\end{aligned} \tag{5.30}$$

This contradiction yields the lemma for n being odd.

Case 2. $n = 2k$ is even.

In this case, we have by (4.5)

$$B(n, 1) = \frac{-n}{2(n-1)} = \frac{-k}{2k-1}. \quad (5.31)$$

By the proof of Theorem 4.1 of [LoZ] (Theorem 11.1.1 of [Lon4]), we may further assume $N = (2k-1)s$ for some $m \in \mathbf{N}$.

Thus by (5.19), (5.22) and (5.31), we have

$$M_0 - M_1 + M_2 - \cdots + (-1)^{2N+n-2} M_{2N+n-2} = -2sk. \quad (5.32)$$

On the other hand, we have by (2.9)

$$\begin{aligned} & b_0 - b_1 + b_2 - \cdots + (-1)^{2N+n-2} b_{2N+n-2} \\ = & -b_{2k-1} - (b_{2k+1} + \cdots + b_{6k-3} + \cdots + b_{(s-1)(4k-2)+2k+1} + \cdots + b_{s(4k-2)+2k-1}) \\ & + b_{s(4k-2)+2k-1} \\ = & -1 - s(2k-2+2) + 2 \\ = & -2sk + 1. \end{aligned} \quad (5.33)$$

In fact, we cut off the sequence $\{b_{2k+1}, \dots, b_{s(4k-2)+2k-1}\}$ into s pieces, each of them contains $2k-1$ terms. Moreover, each piece contain 1 for $2k-2$ times and 2 for one time. Thus (5.33) holds.

Now by (5.32), (5.33) and Theorem 2.3, we have

$$\begin{aligned} -2sk &= M_{2N+n-2} - M_{2N+n-3} + \cdots + M_1 - M_0 \\ &\geq b_{2N+n-2} - b_{2N+n-3} + \cdots + b_1 - b_0 \\ &= -2sk + 1. \end{aligned} \quad (5.34)$$

This contradiction yields the lemma for n being even. ■

Lemma 5.2 *We have*

$$i(c_j^{2m_j-2}) < 2N - (n-1), \quad (5.35)$$

for $1 \leq j \leq p$.

Proof. By (5.3) and (5.8), if $i(c_j) > n-1$, then (5.35) holds. Thus it remains to consider the case $i(c_j) = n-1$. By (3.5) and (5.2), we have

$$\begin{aligned} \hat{i}(c_j) &= i(c_j) + p_- + p_0 - r + \sum_{i=1}^r \frac{\theta_i}{\pi} \\ &= i(c_j) - r + \sum_{i=1}^r \frac{\theta_i}{\pi} > n-1, \end{aligned} \quad (5.36)$$

where the second equality follows from $p_- = 0 = p_0$, which holds since c_j is non-degenerate. Plugging $i(c_j) = n - 1$ into (5.36) yields

$$\sum_{i=1}^r \left(\frac{\theta_i}{\pi} - 1 \right) > 0. \quad (5.37)$$

Hence we can write

$$P_{c_j} = R(\theta) \diamond M, \quad (5.38)$$

for some $\theta \in (\pi, 2\pi)$ and $M \in Sp(2n - 4)$. Thus by Theorem 3.1 and the assumption that c_j is non-degenerate, we have

$$\begin{aligned} i(c_j^m) &= m(i(c_j) - r) + 2 \sum_{i=1}^r \mathcal{E} \left(\frac{m\theta_i}{2\pi} \right) - r \\ &= 2\mathcal{E} \left(\frac{m\theta}{2\pi} \right) - 1 + i(\gamma, m), \end{aligned} \quad (5.39)$$

where $\gamma \in \{\xi \in C([0, \tau], Sp(2n - 4)) \mid \xi(0) = I\}$ satisfies $\gamma(\tau) = M$ and $i(\gamma, 1) = n - 2$. The second equality follows from the Symplectic additivity of the index function, cf. Theorem 6.2.7 of [Lon4]. Note that it follows from Theorem 2.2 of [LoZ] (Theorem 10.2.3 of [Lon4]) that

$$i(\gamma, m + 1) - i(\gamma, m) \geq i(\gamma, 1) - \frac{e(M)}{2} \geq 0, \quad \forall m \in \mathbf{N}. \quad (5.40)$$

Here the last inequality holds from $i(\gamma, 1) = n - 2$ and the fact that $e(M) \leq 2(n - 2)$. By (5.3) and (5.8), in order to prove (5.35), it is sufficient to prove

$$i(c_j^{2m_j-2}) < i(c_j^{2m_j-1}). \quad (5.41)$$

By (5.39) and (5.40), in order to prove (5.41), it is sufficient to prove

$$\mathcal{E} \left(\frac{(2m_j - 2)\theta}{2\pi} \right) < \mathcal{E} \left(\frac{(2m_j - 1)\theta}{2\pi} \right). \quad (5.42)$$

In order to satisfy (5.42), it is sufficient to choose

$$\delta < \min \left\{ \frac{\theta}{\pi} - 1, 1 - \frac{\theta}{2\pi} \right\}, \quad (5.43)$$

where θ is given by (5.9). This proves the lemma ■

Proof of Theorem 1.2. Note that by Theorem 2.3, we have

$$M_q \equiv M_q(\overline{\Lambda}, \overline{\Lambda}^0) = \sum_{m \geq 1, 1 \leq j \leq p} \text{rank} \overline{C}_q(E, c_j^m). \quad (5.44)$$

The proof contains three steps:

Step 1. We have $p \geq 2[\frac{n+1}{2}] - 2$.

We consider the following two cases according to the parity of n .

Case 1.1. $n = 2k + 1$ is odd.

In this case, as in Case 1 of Lemma 5.1, we may assume $N = 2ks$ for some $s \in \mathbf{N}$. Then by Theorem 2.2, we have

$$b_{2N-(n-1)+2m} = 1, \quad 1 \leq m < k, \quad k < m \leq n-2; \quad b_{2N} = 2. \quad (5.45)$$

Thus by Theorem 2.3, we have

$$M_{2N-(n-1)+2m} \geq b_{2N-(n-1)+2m} = 1, \quad 1 \leq m < k, \quad k < m \leq n-2; \quad (5.46)$$

$$M_{2N} \geq b_{2N} = 2. \quad (5.47)$$

By Proposition 2.1 and (5.1), (5.7) and (5.8), we have

$$\begin{aligned} M_{2N-(n-1)+2m} &= \sum_{1 \leq j \leq p} \text{rank} \overline{C}_{2N-(n-1)+2m}(E, c_j^{2m_j}) \\ &= \#\{j | i(c_j^{2m_j}) - i(c_j) \in 2\mathbf{Z}, i(c_j^{2m_j}) = 2N - (n-1) + 2m\}, \end{aligned} \quad (5.48)$$

for $1 \leq m \leq n-2$. Hence we have $p \geq n-1$ by (5.45)-(5.48) and Proposition 2.1. In fact, only the $2m_j$ -th iteration $c_j^{2m_j}$ of c_j contribute at most 1 to

$$\sum_{1 \leq m \leq n-2} M_{2N-(n-1)+2m} \geq \sum_{1 \leq m \leq n-2} b_{2N-(n-1)+2m} = n-3+2 = n-1$$

for each $1 \leq j \leq p$. This yields Step 1 for n being odd.

Case 1.2. $n = 2k$ is even.

In this case, as in Case 2 of Lemma 5.1, we may assume $N = (2k-1)s$ for some $s \in \mathbf{N}$. Then by Theorem 2.2, we have

$$b_{2N-(n-1)+2m} = 1, \quad 1 \leq m \leq n-2. \quad (5.49)$$

Thus by Theorem 2.3, we have

$$M_{2N-(n-1)+2m} \geq b_{2N-(n-1)+2m} = 1, \quad 1 \leq m \leq n-2. \quad (5.50)$$

By Proposition 2.1 and (5.1), (5.7) and (5.8), we have

$$\begin{aligned} M_{2N-(n-1)+2m} &= \sum_{1 \leq j \leq p} \text{rank} \overline{C}_{2N-(n-1)+2m}(E, c_j^{2m_j}) \\ &= \#\{j | i(c_j^{2m_j}) - i(c_j) \in 2\mathbf{Z}, i(c_j^{2m_j}) = 2N - (n-1) + 2m\}, \end{aligned} \quad (5.51)$$

for $1 \leq m \leq n - 2$. Hence as in Case 1, we have $p \geq n - 2$ by (5.49)-(5.51) and Proposition 2.1. This yields Step 1 for n being even.

Step 2. We have $p \geq 2\lfloor \frac{n+1}{2} \rfloor - 1$.

Denote the $2\lfloor \frac{n+1}{2} \rfloor - 2$ prime closed geodesics obtained in Step 1 by $\{c_{j_1}, \dots, c_{j_{2\lfloor \frac{n+1}{2} \rfloor - 2}}\}$. Then by (5.48), (5.51) and Proposition 2.1, we have

$$i(c_{j_k}^{2m_{j_k}}) = 2N - (n - 1) + 2\tau_{j_k}, \quad \overline{C}_{2N - (n-1) + 2\tau_{j_k}}(E, c_{j_k}^{2m_{j_k}}) \neq 0, \quad (5.52)$$

for some $1 \leq \tau_{j_k} \leq n - 2$ and $1 \leq k \leq 2\lfloor \frac{n+1}{2} \rfloor - 2$. On the other hand, by Theorem 5.1, there exists a closed geodesic c_{j_0} such that

$$i(c_{j_0}^{2m_{j_0}}) = 2N + (n - 1), \quad \overline{C}_{2N + n - 1}(E, c_{j_0}^{2m_{j_0}}) \neq 0. \quad (5.53)$$

Hence we have $c_{j_0} \notin \{c_{j_1}, \dots, c_{j_{2\lfloor \frac{n+1}{2} \rfloor - 2}}\}$ by (5.52) and (5.53). This yields Step 2.

Step 3. We have $p \geq 2\lfloor \frac{n+1}{2} \rfloor$.

Denote the $2\lfloor \frac{n+1}{2} \rfloor - 1$ prime closed geodesics obtained in Steps 1 and 2 by $\{c_{j_0}, \dots, c_{j_{2\lfloor \frac{n+1}{2} \rfloor - 1}}\}$.

By Theorems 2.2 and 2.3, we have

$$M_{n-1} = \sum_{1 \leq j \leq p, m \geq 1} \text{rank} \overline{C}_{n-1}(E, c_j^m) \geq b_{n-1} = 1. \quad (5.54)$$

Thus it follows from (5.1) and (5.3) that there exist at least one closed geodesic c_j such that $i(c_j) = n - 1$. We have the following two cases:

Case 3.1. We have $\#\{j | i(c_j) = n - 1\} = 1$, i.e., there is only one prime closed geodesic which has index $n - 1$.

Denote the prime closed geodesic which has index $n - 1$ by c_* . Then we have

$$i(c_l) > n - 1, \quad l \in \{1, \dots, p\} \setminus \{*\}. \quad (5.55)$$

Thus it follows from (5.8) that

$$i(c_l^{2m_l - m}) < 2N - (n - 1), \quad \forall m \in \mathbf{N}, l \in \{1, \dots, p\} \setminus \{*\} \quad (5.56)$$

By Lemma 5.2 and (5.3), we have

$$i(c_*^{2m_* - m}) < 2N - (n - 1), \quad \forall m \geq 2. \quad (5.57)$$

Then by (5.1), (5.7), (5.52), (5.53), (5.56), (5.57) and Proposition 2.1, we have

$$\sum_{0 \leq k \leq 2\lfloor \frac{n+1}{2} \rfloor - 2, m \geq 1} \text{rank} \overline{C}_{2N - (n-1)}(E, c_{j_k}^m) \leq 1. \quad (5.58)$$

In fact, the only possible non-zero term is $\text{rank} \overline{C}_{2N-(n-1)}(E, c_*^{2m_*-1})$.

By Theorems 2.2 and 2.3, we have

$$M_{2N-(n-1)} = \sum_{1 \leq j \leq p, m \geq 1} \text{rank} \overline{C}_{2N-(n-1)}(E, c_j^m) \geq b_{2N-(n-1)} = 2. \quad (5.59)$$

Hence there must be another closed geodesic $c_* \notin \{c_{j_0}, \dots, c_{j_{2[\frac{n+1}{2}]-2}}\}$ by (5.58) and (5.59). Especially, we have

$$i(c_*^{2m_*}) = 2N - (n - 1) \quad (5.60)$$

by (5.1), (5.7), (5.56) and Proposition 2.1. This yields Case 3.1.

Case 3.2. We have $\#\{j | i(c_j) = n - 1\} > 1$.

By Proposition 2.1 we have

$$M_{n-1} = \sum_{1 \leq j \leq p, m \geq 1} \text{rank} \overline{C}_{n-1}(E, c_j^m) \geq 2 \quad (5.61)$$

By (5.1), Proposition 2.1, Theorems 2.2 and 2.3 we have

$$\begin{aligned} M_n - M_{n-1} &= M_n - M_{n-1} + \dots + (-1)^n M_0 \\ &\geq b_n - b_{n-1} + \dots + (-1)^n b_0 = b_n - b_{n-1} = -1. \end{aligned} \quad (5.62)$$

Thus we have

$$M_n = \sum_{1 \leq j \leq p, m \geq 1} \text{rank} \overline{C}_n(E, c_j^m) \geq 1. \quad (5.63)$$

Then we must have a prime closed geodesic c_* with

$$i(c_*) = n. \quad (5.64)$$

In fact, by (5.63), we have

$$\overline{C}_n(E, c_*^m) \neq 0. \quad (5.65)$$

for some $1 \leq \star \leq p$ and $m \in \mathbf{N}$. By (5.1) and (5.3), if $i(c_*) \neq n$, we must have $i(c_*) = n - 1$. Thus it follows from Proposition 2.1 that $\overline{C}_n(E, c_*^m) = 0$ for any $m \in \mathbf{N}$. This contradicts to (5.65) and yields (5.64). Now it follows from Proposition 2.1 and (5.64) that

$$\overline{C}_{2N-(n-1)+2m}(E, c_*^m) = 0, \quad \forall m \in \mathbf{Z}. \quad (5.66)$$

Thus we have $c_* \notin \{c_{j_0}, \dots, c_{j_{2[\frac{n+1}{2}]-2}}\}$ by (5.52) and (5.53). This yields Case 3.2.

The proof of Theorem 1.2 is complete. ■

Proof of Theorem 1.3. We have two cases according to Step 3 of the above proof.

Case 1. *If Case 3.1 holds.*

In this case, the closed geodesic c_{j_0} is elliptic by (5.5) and (5.53). While the closed geodesic c_* is elliptic by (5.4) and (5.60).

Case 2. *If Case 3.2 holds.*

In this case, we can find two prime closed geodesics c_{k_1} and c_{k_2} such that $i(c_{k_1}) = n - 1 = i(c_{k_2})$. Then by the proof of Theorem 5 in [Rad4], both c_{k_1} and c_{k_2} are elliptic.

The proof of Theorem 1.3 is complete.

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